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Decision-implementation complexity of cooperative game systems

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Abstract This paper focuses on the decision-implementation complexity (DIC) of cooperative games. The complexity of a control law is a fundamental issue in practice because it is closely related to control cost. First, we formulate implementation measures of strategies as system control protocols. Then, for a class of cooperative games, a decision-implementation system model is established, and an energy-based DIC index is given as the energy expectation under Nash equilibrium strategies. A definition of DIC is presented to describe the optimal values of the DIC index. DIC is calculated by the exhaust algorithm in some specific cases, whereas the one for general cases is too complex to be calculated. In order to obtain a general calculation method, the problem is described in the form of matrices; an analytical expression describing DIC is obtained by using the properties of matrix singular values. Furthermore, when only partial information of actions is shared among players, DIC can be reduced and an improved protocol can be designed as a two-phase protocol. A numerical example is given to show that the DIC obtained in this study is the same as the one obtained by the exhaust algorithm, and that the calculation complexity of the proposed algorithm is much lower.

Keywords cooperative game, distributed protocol, optimal control, decision-implementation complexity, two-phase protocol

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1 Introduction

In practice, the complexity of a control law is a fundamental issue, which is closely related to control cost. This study concerns decision-implementation complexity (DIC) problems for cooperative games, which fall within the field of game theory. Game theory mainly focuses on the interaction between formulated incentive structures, and uses mathematical methods to study phenomena regarding struggle or competitive characters. At present, game theory has wide applications in cell biology [1], the Hercynian economic development [2], international business research [3], computer mediated communication [4], political competition with restrictions [5], auction mechanism in military logistics [6] and many other disciplines [7–10].

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In game problems, decision plays an important role [11, 12] and consists of two parts: decision-making and decision-implementation. Decision-making requires finding optimal or suboptimal strategies from admissible strategies under given game objectives. Decision-implementation aims to design measures to implement these strategies. In order to achieve game objectives, decision-implementation must be considered, since for optimal/suboptimal strategies found in decision-making, their optimal implementation measures might have different costs. Hence, decision-making must take decision-implementation into account, and decision-implementation depends on decision-making. They are mutually coupled.

With respect to decision-making, there are some nice results, including researches on Nash equilibrium such as [13–15]. Decision-making complexity (DMC) has an important effect on game systems, which is determined by the time and space complexity of decision-making algorithms, among other factors [16]. Correspondingly, with respect to decision-implementation, DIC can be described by the minimum cost of all the possible implementation measures, which might be the amount of data, energy, and capital.

In practice, decision complexity (DC) determines the selection of strategies. In many cases, game players look for an optimal strategy which, however, may not be unique nor easy to get; therefore, choosing the most appropriate strategy from a variety of optimal or suboptimal strategies becomes the key to decision, in which the selection criteria depends on DC. Since decision includes two parts: decision-making and decision-implementation, DC should also be determined by two parts: DMC and DIC. To this end, we introduce a definition of DC as follows.

Definition 1. In game problems, DC (denoted by C) is defined as the product of DMC and DIC:

$$C = C_{\mathrm{M}} \cdot C_{\mathrm{I}}, \tag{1}$$

where C_{M} and C_{I} denote DMC and DIC, respectively.

Remark 1. C_{M} and C_{I} are both determined by a variety of factors. As already mentioned, factors determining C_{M} can be time complexity and space complexity of decision-making algorithms; factors that determine C_{I} can be costs of data acquisition and information processing, energy and capital consumed during the implementation.

Remark 2. In (1), a form of product of C_{M} and C_{I} is obtained mainly because decision-making and decision-implementation are mutually coupled in the whole decision activity, decision-implementation depends on decision-making, and decision-making must, in turn, take decision-implementation into account.

Even DIC is as important as DMC in calculating DC. Research in this area is relatively deficient, mainly due to the lack of a quantitative mathematical description and a proper research framework.

This paper aims to study the DIC of games from the perspective of energy, i.e., to measure the DIC of games by energy cost. To this end, we focus on cooperative games since they are a class of problems with very wide practical applications [17–19] among game problems, and more importantly, the theoretical results on cooperative games are richer than non-cooperative games [20]. Cooperative games are also known as positive-sum games, referring to a kind of game which makes the interests of all players increase, or at least one player's interests increase while the interests of the others are not harmed; therefore, the whole system's interests increase.

Subsequently, the energy-based DIC index is related to the energy of control measures and the probability distribution of strategies at Nash equilibrium points. There are many studies on control energy issues in the field of communication. Communication complexity plays a vital role in distributed computing. This concept was first proposed in [21], and then, attracted wide attentions [22–25]. Wong and Baillieul introduced the concept of communication complexity to control systems for the first time, and presented a strict definition of communication complexity in control systems as control communication complexity in [26, 27]. Subsequently, they extended the concept of control communication complexity to nonlinear systems in [28], and proposed the concept of control energy complexity, and control time complexity. Control energy complexity is similar to control communication complexity, and they are both indexes by which the costs of control protocols are measured. These theoretical results have wide application prospects (e.g., [29]). However, energy in the field of communication does not contain strategies and implementation measures in cooperative games.

In this study, the main issue was that general systems theory could not be used. The competitive character of games determines the no-sharing (privacy) character of strategies, which induces that communication control models and methods of general distributed multi-agent systems cannot be applied to this problem directly. It is also difficult to present the problem in an appropriate form for optimizing the energy-based index.

The main method in this paper consists of formulating strategy implementation measures as system control protocols, via establishing a decision-implementation system model of cooperative games and using system control theory to optimize the energy-based index with the help of the properties of matrix singular values, in order to get the optimal index value, namely DIC.

The main contributions of this paper can be summarized as (1) formulate the implementation measures of strategies as system control protocols, establish a decision-implementation system model of cooperative games, present an energy-based index as the expectation of the energy under Nash equilibrium strategies, and present a definition of DIC as the optimal values of the energy-based index, so that a quantitative mathematical description and a research framework for DIC is constructed; (2) get an analytical expression (a general calculation method) of DIC under the decision-implementation system model of cooperative games; (3) design an improved two-phase protocol to reduce DIC with partial information of actions shared between the players.

The rest of this paper is organized as follows. In Section 2, the model of decision-implementation systems is formulated, and a calculation example of DIC is given. In Section 3, an analytical expression of DIC is given in terms of matrix singular values. In Section 4, a quantitative result of DIC is presented and proved. In Section 5, a kind of improved control protocol — the two-phase protocol is designed to reduce DIC. In Section 6, the example in Section 2 is used to show the efficiency of the main result of this paper. In Section 7, we conclude this paper and describe future work.

2 Problem formulation

In this section, we provide a mathematical formulation for a decision-implementation system of cooperative games, an energy-based index and a definition of DIC. We illustrate the actual background of DIC and provide the method to calculate it under a specific circumstance. An example is given to show the calculation of DIC.

2.1 System model

Consider the decision-implementation system of a cooperative game with two players A and B,

$$\begin{cases} \boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k, u_k, v_k), \\ \boldsymbol{y}_k^A = \boldsymbol{h}_k^A(\boldsymbol{x}_k), \\ \boldsymbol{y}_k^B = \boldsymbol{h}_k^B(\boldsymbol{x}_k), \\ u_k = P_k^A(Q_k^A(\boldsymbol{y}_k^A), \ldots, Q_1^A(\boldsymbol{y}_1^A), \alpha), \\ v_k = P_k^B(Q_k^B(\boldsymbol{y}_k^B), \ldots, Q_1^B(\boldsymbol{y}_1^B), \beta), \\ z_k = c(\boldsymbol{x}_k), \end{cases} \tag{2}$$

where $\boldsymbol{x}_k$ is an N-dimensional state vector, $\boldsymbol{f}$ is a function of the cooperative game, $\boldsymbol{y}_k^A, \boldsymbol{y}_k^B$ is information obtained by Player A and Player B, respectively, and Q_k^A, Q_k^B are quantization/coding functions. $\alpha \in \mathcal{A}$ is a finite action set of Player A, denoted by $\mathcal{A} = \{1, \ldots, m\}$; similarly, $\beta \in \mathcal{B} = \{1, \ldots, n\}$ is a finite action set of Player B. α takes i in $\mathcal{A}$ with probability distribution p_i, satisfying $\sum_{i=1}^m p_i = 1$; β takes j in $\mathcal{B}$ with probability distribution q_j, satisfying $\sum_{j=1}^n q_j = 1$. Denote $\sigma_A^* = \{p_1, \ldots, p_m\}, \sigma_B^* = \{q_1, \ldots, q_n\}$, and (σ_A^*, σ_B^*) is a mixed strategy Nash equilibrium point of the game. Once selected, Strategy (α, β) remains unchanged. Accordingly, u_k, v_k are scalar control measures, which A and B take for Strategy (α, β), respectively. z_k is the output of the system, and c is a scalar output function.

Model (2) formulates the main object of this paper.

Let $\boldsymbol{H} = [H_{ij}]$ be the game target matrix of order $m \times n$. $\boldsymbol{H}$ is called realizable if, at the terminal time T, for any $\alpha = i, \beta = j$, there exists $(u_k(i), v_k(j))$ such that

$$c(\boldsymbol{x}_T) = H_{ij}. \tag{3}$$

From a game perspective, we need to assume that there are no direct communication channels between Players A and B; that is, they cannot communicate with each other directly. Hence, the game players keep their respective actions secret from each other; thereby, they both draw measures up to implement independently. According to this, we introduce a definition of DIC as follows.

For $\mathcal{U} = \{u(1), \ldots, u(m)\}$, $\mathcal{V} = \{v(1), \ldots, v(n)\}$, $u(i)$ is the control strategy corresponding to Action $\alpha = i$ and $v(j)$ is the control strategy corresponding to Action $\beta = j$, and the control energy cost that the control consumes is

$$I(\mathcal{U}, \mathcal{V}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \left(\int_0^T u^2(i)\mathrm{dt} + \int_0^T v^2(j)\mathrm{dt} \right) = \sum_{i=1}^{m} p_i \int_0^T u^2(i)\mathrm{dt} + \sum_{j=1}^{n} q_j \int_0^T v^2(j)\mathrm{dt}. \tag{4}$$

Definition 2. Decision-implementation complexity (DIC) is defined as the optimal value of (4); namely the lowest energy cost.

Remark 3. Control protocols satisfying the constraint (3) exist in most cases; that is, the feasible region is non-empty (see Lemma 1).

The objective is to find the protocols, which consume the lowest energy cost in the feasible region, and the lowest energy cost is DIC. Actually, this is an optimization problem taking (4) as the index and (3) as the constraint.

Remark 4. If the dimension of the target matrix is infinite, that is, if m, n, the numbers of the two players' actions, are infinite, formula (4) becomes

$$I(\mathcal{U}, \mathcal{V}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i q_j \left(\int_0^T u^2(i)\mathrm{dt} + \int_0^T v^2(j)\mathrm{dt} \right) = \sum_{i=1}^{\infty} p_i \int_0^T u^2(i)\mathrm{dt} + \sum_{j=1}^{\infty} q_j \int_0^T v^2(j)\mathrm{dt}. \tag{5}$$

Clearly, the optimization of (5) is much more difficult than (4); therefore, for simplicity, we mainly deal with (4) in this paper.

Remark 5. The controls u_k, v_k in the issue discussed in this paper are assumed to be scalars, in order to facilitate the discussion. They can be generalized to vector functions and the analysis methods and results are similar.

In the next subsection, we will illustrate the DIC of a cooperative game problem.

2.2 DIC of a data acquisition cooperative game system

Suppose that there are two densely populated regions, A and B, in a hot-weather country that is short of power. In order to help the residents in these two regions relieve summer-heat, the electricity sector requires an efficient electricity supply scheme. It has been decided that the supply scheme is drawn up based on the sample data of the land surface temperatures of the two regions.

Assume that the sensor networks [30] in the two regions both hope to store the sample data of their respective regions in the central database as much as possible, so that the supply scheme drawn up by the electricity sector is beneficial to their respective regions. The memory size of the central database is fixed and the amount of data of only one region cannot meet the data input requirement. One can view this as the formation of a cooperative game.

The main way for sensor nodes to save energy is the sleeping mechanism; that is, they close their wireless communication modules, data acquisition modules and even the calculation module in order to save energy when they are not occupied in sensing tasks or forwarding sensing data to other nodes. Accordingly, there are two main kinds of work actions for the nodes in the sensor networks, in these two regions, as follows.

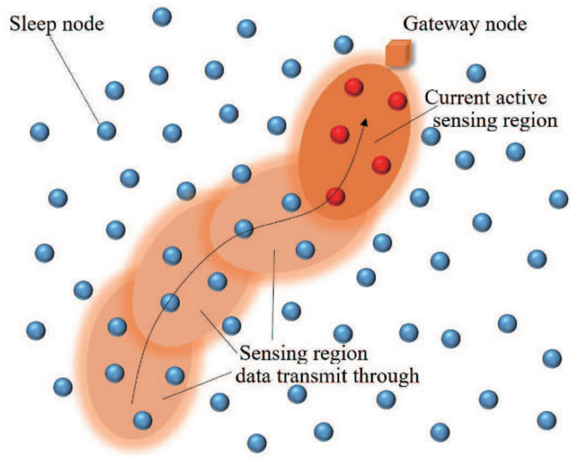

Figure 1 (Color online) Movement of active sensing region in a sensor network.

① When a sensing task occurs, only the sensor nodes in the adjacent region are in working states, which forms an active sensing region. The active sensing region moves with data transmission to the gateway node, so that, in order to save energy, the original working nodes can turn to sleep after leaving the active sensing region, as shown in Figure 1.

② Power is supplied to the main routing nodes all the time. These nodes are responsible for transmitting sensing data to other nodes. That is, the main routing nodes are always in working states. The rest of the nodes' sleeping mechanism remains unchanged, just like Action ①. Thereby, data transfer efficiency is improved with more energy consumption.

At time k, the work policy (whether to work in the next time slot) of the nodes in the sensor network in Region A is denoted by $\boldsymbol{u}_k$, and the work policy of the nodes in the sensor network in Region B is denoted by $\boldsymbol{v}_k$. In Action ①, all the nodes can turn to work or sleep; for example, every element of $\boldsymbol{u}_k$ can be 0 or 1. However, in Action ②, the main routing nodes can only be working and the corresponding elements of $\boldsymbol{u}_k$ can only be 1.

Sensor networks in these two regions select respective work actions based on the Nash equilibrium points of the game. The work action of Region A is denoted by α, and the work action of Region B is denoted by β. Once selected, the work action of the sensor network in each region remains unchanged and is kept as private information, which is unknown to the sensor network in the other region during the entire data acquisition.

The sensor network in Region A consists of n_A sensor nodes, and the sensor network in Region B consists of n_B sensor nodes. The sensor nodes in each region form a multi-hop network autonomously, and are responsible for acquiring and processing data. The nodes in the edge of the sensor networks have to send data to the gateway nodes through other nodes (routing nodes). The gateway nodes are responsible for collecting data transmitted by the sensor nodes. The gateway nodes are connected to the upper transmission network, which provides the communication bandwidth and reliability between the gateway nodes and the base station. The transmission network is connected to the Internet through the base station, which is responsible for collecting and forwarding all the data sent by the transmission network to the Internet. Finally, data acquired by the sensor nodes are sent, via the Internet, to the central database for processing and storage. The amount of data stored at time k is x_k. The central database provides remote data services; the data analysis program can use them through clients connected to the Internet. The whole process is shown in Figure 3.

The relationship between the current amount of data stored in the central database at the current time, the continuous work time of the sensor nodes and the amount of stored data at the next time is denoted by f; then, f is determined by the whole data acquisition system and we obtain the following model:

$$x_{k+1} = f(x_k, \boldsymbol{u}_k, \boldsymbol{v}_k). \tag{6}$$

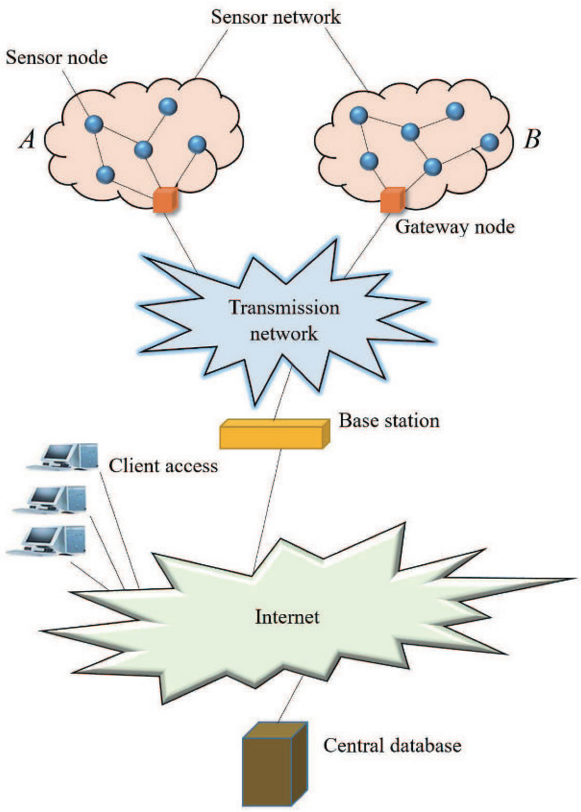

Figure 2 (Color online) The whole data transmission process.

If we use y_k^A to represent the amount of data acquired from Region A from the total amount of data at time k, y_k^B to represent the amount of data acquired from Region B from the total amount of data at time k, and denote their proportions of total amount of data by h_k^A and h_k^B, respectively, then we obtain the following relationship:

$$\begin{cases} y_k^A = h_k^A(x_k), \\ y_k^B = h_k^B(x_k). \end{cases} \tag{7}$$

Furthermore, if at time k, y_k^A is quantified by the quantization/coding function Q_k^A and sent to the sensor network in Region A through the Internet, y_k^B is quantified by the quantization/coding function Q_k^B and sent to the sensor network in Region B, the sensor networks in these two regions use the historical amount of data and programs $\boldsymbol{P}_k^A, \boldsymbol{P}_k^B$ are set according to work strategy ($\alpha = ⓘ, \beta = ⓙ$) in order to adjust the sensor nodes' continuous work time, respectively. Then $\boldsymbol{u}_k$ and $\boldsymbol{v}_k$ can be written as follows:

$$\begin{cases} \boldsymbol{u}_k = \boldsymbol{P}_k^A(Q_k^A(y_k^A), \ldots, Q_1^A(y_1^A), \alpha), \\ \boldsymbol{v}_k = \boldsymbol{P}_k^B(Q_k^B(y_k^B), \ldots, Q_1^B(y_1^B), \beta). \end{cases} \tag{8}$$

Let c represent an index function of whether the amount of sample data meets the research requirement, that is

$$z_k = c(x_k). \tag{9}$$

Time T is the terminal time of data acquisition, if the function value z_T takes 1 at time T, which is evident that the amount of land surface temperature data acquired by sensor networks in the two regions meets the requirement; that is,

$$z_T = H_{ij} = 1, \tag{10}$$

where $\boldsymbol{H} = [H_{ij}]$ is the data acquisition target matrix of order 2×2, and the data acquisition target corresponding to strategy ($\alpha = ⓘ, \beta = ⓙ$) is H_{ij}.

B

A	Action	①	②
	①	$(4,4)$	$(2,6)$
	②	$(6,2)$	$(1,1)$

Figure 3 Payoff matrix.

Thus, the whole data acquisition can be described by the following model:

$$\begin{cases} x_{k+1} = f(x_k, \boldsymbol{u}_k, \boldsymbol{v}_k), \\ y_k^A = h_k^A(x_k), \\ y_k^B = h_k^B(x_k), \\ \boldsymbol{u}_k = \boldsymbol{P}_k^A(Q_k^A(y_k^A), \ldots, Q_1^A(y_1^A), \alpha), \\ \boldsymbol{v}_k = \boldsymbol{P}_k^B(Q_k^B(y_k^B), \ldots, Q_1^B(y_1^B), \beta), \\ z_k = c(x_k), \end{cases} \tag{11}$$

where $\alpha \in \mathcal{A} = \{①, ②\}$ is the work action set of the sensor network in Region A, and $\beta \in \mathcal{B} = \{①, ②\}$ is the work action set of the sensor network in Region B. p_i is the probability distribution when α takes ⓘ in $\mathcal{A}$, satisfying $p_1 + p_2 = 1$; q_j is the probability distribution when β takes ⓙ in $\mathcal{B}$, satisfying $q_1 + q_2 = 1$. Denote $\sigma_A^* = \{p_1, p_2\}, \sigma_B^* = \{q_1, q_2\}$, and (σ_A^*, σ_B^*) is a mixed strategy Nash equilibrium point of the game.

Actually, the system (11) can be regarded as a system (2) with conversions of scalar and vector functions; therefore, the analysis methods are similar to each other.

The control strategy corresponding to the work action $\alpha =$ ⓘ of the sensor network is denoted by $\{\boldsymbol{u}_k(ⓘ)\}_{k=0}^{T-1}$, the control strategy corresponding to the work action $\beta =$ ⓙ of the sensor network is denoted by $\{\boldsymbol{v}_k(ⓙ)\}_{k=0}^{T-1}$, $\mathcal{U} = \{\{\boldsymbol{u}_k(①)\}_{k=0}^{T-1}, \{\boldsymbol{u}_k(②)\}_{k=0}^{T-1}\}$, and $\mathcal{V} = \{\{\boldsymbol{v}_k(①)\}_{k=0}^{T-1}, \{\boldsymbol{v}_k(②)\}_{k=0}^{T-1}\}$. Then, the work time of the sensor nodes in the protocol is

$$\begin{aligned} I(\mathcal{U}, \mathcal{V}) &= \sum_{i=1}^{2}\sum_{j=1}^{2} p_i q_j \left(\sum_{k=0}^{T-1} \boldsymbol{u}_k(ⓘ)' \boldsymbol{u}_k(ⓘ) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(ⓙ)' \boldsymbol{v}_k(ⓙ) \right) \\ &= \sum_{i=1}^{2} p_i \sum_{k=0}^{T-1} \boldsymbol{u}_k(ⓘ)' \boldsymbol{u}_k(ⓘ) + \sum_{j=1}^{2} q_j \sum_{k=0}^{T-1} \boldsymbol{v}_k(ⓙ)' \boldsymbol{v}_k(ⓙ). \end{aligned} \tag{12}$$

This is a working time cost index of the sensor network. Since the energy consumption of sensor networks is closely related to time, this index also reflects the energy cost of the network.

The objective is to find the protocols with the lowest energy cost for the network, under the premise of ensuring that the amount of sample data stored in the central database meets the research requirement. Since the energy cost is essentially a quantitative index of controls, this lowest energy cost is the DIC of the data acquisition game system's Nash equilibrium point.

2.3 Calculation of DIC

We now calculate the DIC for the example in Subsection 2.2 in detail. We assume that the payoff matrix of the game in the example is as Figure 3.

Remark 6. The payoff here can be characterized by the data transmission efficiency. The last pair of efficiency (1,1) is due to the networks in the two regions both adopting strategy ② such that the amount of data acquired in each time slot is too large for the base station to store and forward all the data, resulting in data loss and reduction of transmission efficiency.

After a simple calculation, we can show that this game has three Nash equilibrium points: (①,②), (②,①) and (σ^*, σ^*), where $\sigma^* = \{1/3, 2/3\}$. The calculation of (σ^*, σ^*) is given as follows.

Let $p_1 = p,\ p_2 = 1 - p,\ q_1 = q,\ q_2 = 1 - q$,

$$W(p) = (p, 1-p)\begin{pmatrix} 4 & 2 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} q \\ 1-q \end{pmatrix} = -3pq + p + 5q + 1, \tag{13}$$

then

$$\frac{\mathrm{d}W(p)}{\mathrm{d}p} = -3q + 1 = 0 \Rightarrow q = \frac{1}{3}. \tag{14}$$

The calculation of p is similar.

For simplicity, there are only two sensor nodes in the sensor network in each region, which are denoted by a_1, a_2 and b_1, b_2, respectively. The nodes working continuously in strategy ② are a_2, b_2, respectively. Each sensor node acquires 1 bit of data at each working time slot; all the data acquired are valid. The energy consumption and working time are proportional and take the same value. Then, Model (6) can be simplified as

$$f(x_k, \boldsymbol{u}_k, \boldsymbol{v}_k) = x_k + \boldsymbol{u}_k'\boldsymbol{u}_k + \boldsymbol{v}_k'\boldsymbol{v}_k. \tag{15}$$

Programs $\boldsymbol{P}_k^A, \boldsymbol{P}_k^B$ set according to the work strategy ($\alpha =$ ⓘ, $\beta =$ ⓙ) are the same and time-invariant, denoted by $\boldsymbol{P}_k^A = \boldsymbol{P}_k^B = \boldsymbol{P}$, and similarly, $Q_k^A = Q_k^B = Q$. Hence, the effect of $\boldsymbol{P}$ and Q is simplified to

$$\begin{cases} \boldsymbol{u}_k = \boldsymbol{P}(Q(y_k^A), Q(y_{k-1}^A), \alpha), \\ \boldsymbol{v}_k = \boldsymbol{P}(Q(y_k^B), Q(y_{k-1}^B), \beta). \end{cases} \tag{16}$$

In detail, when $\alpha =$ ①,

$$\boldsymbol{u}_k(①) = \boldsymbol{P}(Q(y_k^A), Q(y_{k-1}^A), ①) = \begin{cases} (0,1)', & k \text{ is even}, \\ (1,0)', & k \text{ is odd}. \end{cases} \tag{17}$$

When $\alpha =$ ②,

$$\boldsymbol{u}_k(②) = \boldsymbol{P}(Q(y_k^A), Q(y_{k-1}^A), ②) = \begin{cases} (0,1)', & y_k^A - y_{k-1}^A = 2, \\ (1,1)', & y_k^A - y_{k-1}^A < 2. \end{cases} \tag{18}$$

β is similar to α.

Assume that the amount of sample data needed is 5 bits. Then

$$c(x_k) = I_{\{x_k \geqslant 5\}}. \tag{19}$$

Thus, Eq. (11) can be simplified as

$$\begin{cases} x_{k+1} = x_k + \boldsymbol{u}_k'\boldsymbol{u}_k + \boldsymbol{v}_k'\boldsymbol{v}_k, \\ y_k^A = h_k^A(x_k), \\ y_k^B = h_k^B(x_k), \\ \boldsymbol{u}_k = \boldsymbol{P}(Q(y_k^A), Q(y_{k-1}^A), \alpha), \\ \boldsymbol{v}_k = \boldsymbol{P}(Q(y_k^B), Q(y_{k-1}^B), \beta), \\ z_k = I_{\{x_k \geqslant 5\}}. \end{cases} \tag{20}$$

Without loss of generality, let the initial measures in these two regions be $\boldsymbol{u}_0 = (0,1)', \boldsymbol{v}_0 = (0,1)'$, and the initial amount of data stored in the center database be $x_0 = 0$. Then, for the case of strategy (①,①), Eq. (17) shows that the terminal time is $T = 3$, the total amount of data is $x_3 = 6 \geqslant 5$, and the energy consumed is

$$\sum_{k=0}^{T-1} \boldsymbol{u}_k(①)'\boldsymbol{u}_k(①) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(①)'\boldsymbol{v}_k(①) = 6. \tag{21}$$

For the case of strategy (①,②), Eqs. (17) and (18) show that the terminal time is $T = 2$, the total amount of data is $x_2 = 5 \geqslant 5$, and the energy consumed is

$$\sum_{k=0}^{T-1} \boldsymbol{u}_k(①)'\boldsymbol{u}_k(①) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(②)'\boldsymbol{v}_k(②) = 5. \tag{22}$$

For the case of strategy (②,①), Eqs. (17) and (18) show that the terminal time is $T = 2$, the total amount of data is $x_2 = 5 \geqslant 5$, and the energy consumed is

$$\sum_{k=0}^{T-1} \boldsymbol{u}_k(②)'\boldsymbol{u}_k(②) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(①)'\boldsymbol{v}_k(①) = 5. \tag{23}$$

For the case of strategy (②,②), combing with the analysis in Remark 6, we can get the formula of the consumed energy

$$\sum_{k=0}^{T-1} \boldsymbol{u}_k(②)'\boldsymbol{u}_k(②) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(②)'\boldsymbol{v}_k(②) = 4T - 2. \tag{24}$$

First, consider the Nash equilibrium point (①,②) and regard it as a mixed strategy Nash equilibrium point (σ_A^*, σ_B^*), where $\sigma_A^* = \{1, 0\}, \sigma_B^* = \{0, 1\}$. Using (22), the index (12) becomes

$$I(\mathcal{U}, \mathcal{V}) = \sum_{k=0}^{T-1} \boldsymbol{u}_k(①)'\boldsymbol{u}_k(①) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(②)'\boldsymbol{v}_k(②) = 5. \tag{25}$$

Then, consider the Nash equilibrium point (②,①) and regard it as a mixed strategy Nash equilibrium point (σ_A^*, σ_B^*), where $\sigma_A^* = \{0, 1\}, \sigma_B^* = \{1, 0\}$. Using (23), the index (12) becomes

$$I(\mathcal{U}, \mathcal{V}) = \sum_{k=0}^{T-1} \boldsymbol{u}_k(②)'\boldsymbol{u}_k(②) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(①)'\boldsymbol{v}_k(①) = 5. \tag{26}$$

Finally, consider the mixed strategy Nash equilibrium point(σ^*, σ^*), where $\sigma^* = \{1/3, 2/3\}$. Using (21) and (24), the index (12) becomes

$$\begin{aligned} I(\mathcal{U}, \mathcal{V}) = {} & \frac{1}{3} \cdot \frac{1}{3} \cdot \left(\sum_{k=0}^{T-1} \boldsymbol{u}_k(①)'\boldsymbol{u}_k(①) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(①)'\boldsymbol{v}_k(①) \right) \\ & + \frac{1}{3} \cdot \frac{2}{3} \cdot \left(\sum_{k=0}^{T-1} \boldsymbol{u}_k(①)'\boldsymbol{u}_k(①) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(②)'\boldsymbol{v}_k(②) \right) \\ & + \frac{2}{3} \cdot \frac{1}{3} \cdot \left(\sum_{k=0}^{T-1} \boldsymbol{u}_k(②)'\boldsymbol{u}_k(②) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(①)'\boldsymbol{v}_k(①) \right) \\ & + \frac{2}{3} \cdot \frac{2}{3} \cdot \left(\sum_{k=0}^{T-1} \boldsymbol{u}_k(②)'\boldsymbol{u}_k(②) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(②)'\boldsymbol{v}_k(②) \right) \\ = {} & \frac{1}{9} \cdot 6 + \frac{2}{9} \cdot 5 + \frac{2}{9} \cdot 5 + \frac{4}{9} \cdot (4T - 2) = \frac{16}{9} T + 2. \end{aligned} \tag{27}$$

From the characteristics of actions ① and ②, it can be seen that the protocol consisting of (17) and (18) is optimal. Hence, the index values in (25) to (27) are optimal and DICs of the three Nash equilibrium points, respectively.

It can also be seen that the above method to calculate DIC is essentially a brute-force method and only applicable to the specific case. For the DIC of the general decision-implementation system (2) of cooperative games, we will carry out a quantitative analysis in Section 4.

3 Matrix representation of DIC

Our main method is to describe the original problems (3) and (4) in matrix form and get an analytical expression of DIC based on the properties of matrix singular values.

Let the admissible control set $\mathcal{L}$ be a closed subspace of $L^2[0,T]$, and $\mathcal{L}\otimes\mathcal{L}$ represent the tensor product Hilbert space. At time T, the output mapping $z=c(\boldsymbol{x})$ of the system (2) can be regarded as a functional from $\mathcal{L}\otimes\mathcal{L}$ to the Real line, denoted by Φ. In the rest of this paper, we assume that Φ is a bounded bilinear functional. The constraint (3) becomes

$$\Phi(u(i),v(j))=H_{ij}. \tag{28}$$

Let $\{\xi_1,\xi_2,\dots\}$ and $\{\eta_1,\eta_2,\dots\}$ be orthonormal bases (possibly the same) of $\mathcal{L}$. Let L be the dimension of $\mathcal{L}$, which could be infinity. If $u=\sum_{i=1}^{L} r_i\xi_i, v=\sum_{j=1}^{L} s_j\eta_j$, then, by the bounded bilinearity of $\Phi(\cdot,\cdot)$, we have

$$\Phi(u,v)=\Phi\left(\sum_{i=1}^{L} r_i\xi_i,\sum_{j=1}^{L} s_j\eta_j\right)=\sum_{i,j=1}^{L} r_is_j\Phi(\xi_i,\eta_j). \tag{29}$$

Hence, Φ has a matrix representation

$$\boldsymbol{\Phi}:=\begin{pmatrix}\Phi(\xi_1,\eta_1) & \cdots & \Phi(\xi_1,\eta_L)\\ \vdots & \ddots & \vdots\\ \Phi(\xi_L,\eta_1) & \cdots & \Phi(\xi_L,\eta_L)\end{pmatrix}. \tag{30}$$

The matrix of $(u(1),\dots,u(m))'$ under the standard orthonormal base $\{\xi_1,\xi_2,\dots\}$ is denoted by $\boldsymbol{U}$, namely

$$\boldsymbol{U}=\begin{pmatrix}r(1)_1 & \cdots & r(1)_L\\ \vdots & \ddots & \vdots\\ r(m)_1 & \cdots & r(m)_L\end{pmatrix}, \tag{31}$$

where $u(i)=\sum_{j=1}^{L} r(i)_j\xi_j, i=1,\dots,m$. Similarly, the matrix of $(v(1),\dots,v(n))'$ under the standard orthonormal base $\{\eta_1,\eta_2,\dots\}$ is denoted by $\boldsymbol{V}$; then, the constraint (28) can be written in matrix form

$$\boldsymbol{U\Phi V}'=\boldsymbol{H}. \tag{32}$$

Accordingly, the optimization index (4) can be written as

$$I(\mathcal{U},\mathcal{V})=\mathrm{tr}(\boldsymbol{P}UU'\boldsymbol{P})+\mathrm{tr}(\boldsymbol{Q}VV'\boldsymbol{Q}), \tag{33}$$

where $\boldsymbol{P}=\mathrm{diag}\{\sqrt{p_1},\dots,\sqrt{p_m}\},\boldsymbol{Q}=\mathrm{diag}\{\sqrt{q_1},\dots,\sqrt{q_n}\}$.

We now show that the feasible region of the above optimization problem is not empty.

Lemma 1. There exist control protocols, which can realize the target matrix $\boldsymbol{H}$ (i.e., $\boldsymbol{H}$ is realizable), if and only if the rank of matrix representation $\boldsymbol{\Phi}$ is not less than that of $\boldsymbol{H}$.

Remark 7. Similar to [31], the discussion in this paper is based on the controllability and stability of the system, and the assumption that the control objective (game goal) is realizable; that is, the feasible region of the condition optimization problem is not empty. Lemma 1 shows that this assumption can be satisfied. In fact, most actual systems are able to guarantee that control objectives are realizable, see [32, 33].

The conclusion of the above lemma can hold by the matrices' properties; interested readers can refer to the proof in [31].

The following is the definition of the regular matrix representation.

Definition 3. The matrix representation $\boldsymbol{\Phi}$ is regular, if its rank is s and its l-dimensional leading principal minor is non-zero for $l\leqslant s$.

Definition 4. The matrix representation $\boldsymbol{\Phi}$ is strongly regular, if it is finite dimensional and regular or if it is in diagonal form such that the diagonal elements satisfy the ordering

$$\forall j, \quad \Phi_{j,j} \geqslant \Phi_{j+1,j+1} > 0. \tag{34}$$

The following is the definition of the Generalized Optimization Problem.

Definition 5. Let $\boldsymbol{H}$ be the $m \times n$ target matrix, $\boldsymbol{\Phi}_l$ be the matrix corresponding to the l-dimensional leading principal minor of $\boldsymbol{\Phi}$ with $l \geqslant \max(m, n)$, and x, y be the arbitrary positive integers (not depending on $\boldsymbol{H}$). The Generalized Optimization Problem is defined by

$$I(\mathcal{U}, \mathcal{V}) = \frac{1}{x}\mathrm{tr}(\boldsymbol{U}U') + \frac{1}{y}\mathrm{tr}(\boldsymbol{V}V'), \tag{35}$$

subject to the constraint

$$\boldsymbol{U}\boldsymbol{\Phi}_l\boldsymbol{V}' = \boldsymbol{H}, \tag{36}$$

denoted by $(\boldsymbol{H}, \boldsymbol{\Phi}_l; x, y)$.

Finally, the singular values of the infinite dimensional matrix representation $\boldsymbol{\Phi}$ are defined as follows: arrange the matrix's singular values in descending order $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_k \geqslant \cdots$, and $\boldsymbol{\Phi}_l$ is the matrix corresponding to the l-dimensional leading principal minor of $\boldsymbol{\Phi}$. Then, from [31], for any given i, $\sigma_i(\boldsymbol{\Phi}_l)$ increases monotonically with l, and $\sigma_i(\boldsymbol{\Phi}_l) \leqslant \|\Phi\| < \infty$, $\forall i, l$. Hence, we can define

$$\sigma_i(\boldsymbol{\Phi}) = \lim_{l \to \infty} \sigma_i(\boldsymbol{\Phi}_l) < \infty. \tag{37}$$

4 Quantitative analysis of DIC

Using the method of matrix representation, we can obtain the following DIC characterization.

Theorem 1. Consider the bounded bilinear output mapping Φ of the system (2), with a regular matrix representation $\boldsymbol{\Phi}$ with rank s. Let $\boldsymbol{H}$ be an $m \times n$ target matrix such that $s \geqslant \max(m, n)$. The infimum control cost (DIC) of control protocols, which realize $\boldsymbol{H}$, is given by

$$\hat{C}_\Phi(\boldsymbol{H}) := \min_{\mathcal{U},\mathcal{V} \subset \mathcal{L}} I(\mathcal{U}, \mathcal{V}) = 2 \sum_{k=1}^{\min(m,n)} \frac{\sigma_k(\boldsymbol{P}HQ)}{\sigma_k(\boldsymbol{\Phi})}, \tag{38}$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_k \geqslant \cdots$ are the matrices' singular values. Furthermore, if the matrix representation is strongly regular, there exists a control protocol which achieves this DIC.

Remark 8. The result of Theorem 1 heuristically extends the result of [31]. They are distinguished for that Eq. (38) does not depend on the dimension of $\boldsymbol{H}$; that is, m, n are allowed to tend to infinity, which deals just with the index case (5) in Remark 4.

In order to prove the theorem, we need the following lemma.

Lemma 2. Consider an invertible matrix $\boldsymbol{\Phi}_l$ and an $l \times l$ target matrix $\boldsymbol{H}$ with rank r. The minimum control cost for the Generalized Optimization Problem $(\boldsymbol{H}, \boldsymbol{\Phi}_l; x, y)$ is achievable and is given by the formula

$$\hat{C}_{\boldsymbol{\Phi}_l}(\boldsymbol{H}) = \frac{2}{\sqrt{xy}} \sum_{k=1}^{r} \frac{\sigma_k(\boldsymbol{H})}{\sigma_k(\boldsymbol{\Phi}_l)}. \tag{39}$$

The proof of the above lemma can be referred to in [31]. The proof of Theorem 1 is given below.

Proof. Rewrite the original problem (32) and (33) as follows:

$$I(\mathcal{U}, \mathcal{V}) = \mathrm{tr}(\boldsymbol{P}UU'\boldsymbol{P}) + \mathrm{tr}(\boldsymbol{Q}VV'\boldsymbol{Q}), \tag{40}$$

subject to the constraint

$$\boldsymbol{P}U\boldsymbol{\Phi}\boldsymbol{V}'\boldsymbol{Q} = \boldsymbol{P}HQ. \tag{41}$$

Denote $\widetilde{\boldsymbol{U}} = \boldsymbol{P}U, \widetilde{\boldsymbol{V}} = \boldsymbol{Q}V, \widetilde{\boldsymbol{H}} = \boldsymbol{P}HQ$, and the original problem can be written as

$$I(\widetilde{\mathcal{U}}, \widetilde{\mathcal{V}}) = \mathrm{tr}\left(\widetilde{\boldsymbol{U}}\widetilde{\boldsymbol{U}}'\right) + \mathrm{tr}\left(\widetilde{\boldsymbol{V}}\widetilde{\boldsymbol{V}}'\right), \tag{42}$$

subject to the constraint

$$\widetilde{\boldsymbol{U}}\boldsymbol{\Phi}\widetilde{\boldsymbol{V}}' = \widetilde{\boldsymbol{H}}, \tag{43}$$

where $\widetilde{\mathcal{U}} = \{\sqrt{p_1}u(1), \dots, \sqrt{p_m}u(m)\}, \widetilde{\mathcal{V}} = \{\sqrt{q_1}v(1), \dots, \sqrt{q_n}v(n)\}$. Eqs. (42) and (43) constitute a Generalized Optimization Problem $(\widetilde{\boldsymbol{H}}, \boldsymbol{\Phi}; 1, 1)$.

Consider the Generalized Optimization Problem $(\widetilde{\boldsymbol{H}}, \boldsymbol{\Phi}_s; 1, 1)$, and Lemma 2 shows that the theorem holds when $\boldsymbol{\Phi}$ is finite dimensional. Hence, we assume that $\boldsymbol{\Phi}$ is infinite dimensional.

Let $\widetilde{\mathcal{U}}^* = \{\sqrt{p_1}u^*(1), \dots, \sqrt{p_m}u^*(m)\}, \widetilde{\mathcal{V}}^* = \{\sqrt{q_1}v^*(1), \dots, \sqrt{q_n}v^*(n)\}$ be an optimal solution to the Generalized Optimization Problem $(\widetilde{\boldsymbol{H}}, \boldsymbol{\Phi}; 1, 1)$, with corresponding matrices $\widetilde{\boldsymbol{U}}^*, \widetilde{\boldsymbol{V}}^*$ respectively, satisfying

$$\sqrt{p_i}u^*(i) = \sum_{j=1}^{\infty} r(i)_j \xi_j, \quad \sqrt{q_i}v^*(i) = \sum_{j=1}^{\infty} s(i)_j \eta_j. \tag{44}$$

Define an approximating solution

$$\sqrt{p_i}u^*(i)|_l = \sum_{j=1}^{l} r(i)_j \xi_j, \quad \sqrt{q_i}v_i^*|_l = \sum_{j=1}^{l} s(i)_j \eta_j, \tag{45}$$

with corresponding matrices

$$\widetilde{\boldsymbol{U}}^*|_l = \begin{pmatrix} r(1)_1 & \cdots & r(1)_l \\ \vdots & \ddots & \vdots \\ r(m)_1 & \cdots & r(m)_l \end{pmatrix}, \quad \widetilde{\boldsymbol{V}}^*|_l = \begin{pmatrix} s(1)_1 & \cdots & s(1)_l \\ \vdots & \ddots & \vdots \\ s(n)_1 & \cdots & s(n)_l \end{pmatrix}. \tag{46}$$

Then, the relationship between the control energy of the optimal solution to the Generalized Optimization Problem $(\widetilde{\boldsymbol{H}}, \boldsymbol{\Phi}; 1, 1)$ and that of the approximating solution is as follows:

$$I(\widetilde{\mathcal{U}}^*, \widetilde{\mathcal{V}}^*) = \mathrm{tr}\widetilde{\boldsymbol{U}}^*\widetilde{\boldsymbol{U}}^{*'} + \mathrm{tr}\widetilde{\boldsymbol{V}}^*\widetilde{\boldsymbol{V}}^{*'} = \lim_{l\to\infty}\left(\mathrm{tr}\widetilde{\boldsymbol{U}}^*|_l\widetilde{\boldsymbol{U}}^*|_l' + \mathrm{tr}\widetilde{\boldsymbol{V}}^*|_l\widetilde{\boldsymbol{V}}^*|_l'\right) = \lim_{l\to\infty} I(\widetilde{\mathcal{U}}^*|_l, \widetilde{\mathcal{V}}^*|_l). \tag{47}$$

Define the approximate target matrix

$$\widetilde{\boldsymbol{H}}|_l = [\widetilde{H}_{ij}|_l] = \widetilde{\boldsymbol{U}}^*|_l \boldsymbol{\Phi}_l \widetilde{\boldsymbol{V}}^*|_l', \tag{48}$$

where $\boldsymbol{\Phi}_l$ is the l-th principal minor of $\boldsymbol{\Phi}$. Φ is bounded, therefore, it is continuous, then

$$\widetilde{H}_{ij} = \Phi(\sqrt{p_i}u^*(i), \sqrt{q_j}v^*(j)) = \lim_{l\to\infty} \Phi(\sqrt{p_i}u^*(i)|_l, \sqrt{q_j}v^*(j)|_l) = \lim_{l\to\infty} \widetilde{H}_{ij}|_l, \tag{49}$$

or in a matrix form

$$\widetilde{\boldsymbol{H}} = \lim_{l\to\infty} \widetilde{\boldsymbol{H}}|_l. \tag{50}$$

Since the singular value function is continuous in the space of matrices, we have

$$\sigma_k(\widetilde{\boldsymbol{H}}) = \lim_{l\to\infty} \sigma_k(\widetilde{\boldsymbol{H}}|_l), \quad k = 1, \dots, m. \tag{51}$$

For $\forall l \geqslant \max(m, n)$, define the $l \times l$ matrix

$$\overline{\widetilde{\boldsymbol{H}}|_l} = \begin{pmatrix} \widetilde{\boldsymbol{H}}|_l & \boldsymbol{0}_{m\times(l-n)} \\ \boldsymbol{0}_{(l-m)\times n} & \boldsymbol{0}_{(l-m)\times(l-n)} \end{pmatrix}. \tag{52}$$

It can be seen that $\widetilde{\boldsymbol{H}}|_l$ and $\overline{\widetilde{\boldsymbol{H}}|_l}$ have the same singular values in the corresponding order, the minimum control costs for the Generalized Optimization Problem $(\widetilde{\boldsymbol{H}}|_l, \boldsymbol{\Phi}_l; 1, 1)$ and the Generalized Optimization

Problem $\left(\widetilde{\boldsymbol{H}|_l}, \boldsymbol{\Phi}_l; 1, 1\right)$ are the same. By combining with (47), and considering that $\boldsymbol{\Phi}$ is regular, by Lemma 2 we can obtain

$$\begin{aligned}\hat{C}_{\Phi}(\widetilde{\boldsymbol{H}}) &= I(\widetilde{\mathcal{U}}^*, \widetilde{\mathcal{V}}^*) = \lim_{l\to\infty} I(\widetilde{\mathcal{U}}^*|_l, \widetilde{\mathcal{V}}^*|_l) \geqslant \lim_{l\to\infty} \hat{C}_{\boldsymbol{\Phi}_l}(\widetilde{\boldsymbol{H}}|_l) = \lim_{l\to\infty} \hat{C}_{\boldsymbol{\Phi}_l}\left(\widetilde{\boldsymbol{H}|_l}\right) \\ &= \lim_{l\to\infty} 2\sum_{k=1}^{r} \frac{\sigma_k\left(\widetilde{\boldsymbol{H}|_l}\right)}{\sigma_k(\boldsymbol{\Phi}_l)} = \lim_{l\to\infty} 2\sum_{k=1}^{\min(m,n)} \frac{\sigma_k(\widetilde{\boldsymbol{H}}|_l)}{\sigma_k(\boldsymbol{\Phi}_l)} = 2\sum_{k=1}^{\min(m,n)} \frac{\sigma_k(\widetilde{\boldsymbol{H}})}{\sigma_k(\boldsymbol{\Phi})}.\end{aligned} \tag{53}$$

If $\boldsymbol{\Phi}$ is strongly regular, then, $\sigma_k(\boldsymbol{\Phi}_l) = \sigma_k(\boldsymbol{\Phi})$, $\forall k, l$. Hence, the solution to the Generalized Optimization Problem $(\widetilde{\boldsymbol{H}}|_l, \boldsymbol{\Phi}_l; 1, 1)$ achieves the lower bound in (53).

This completes the proof of Theorem 1.

The analysis in [31] shows that the control energy cost (DIC) can be substantially reduced by using protocols which allow communication between the players. For the extreme scenario where the players have complete prior information of each other's actions, the original optimization problem

$$I(\mathcal{U}, \mathcal{V}) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j \left(\int_0^T u^2(i,j)\mathrm{d}t + \int_0^T v^2(i,j)\mathrm{d}t\right) \tag{54}$$

can be reduced to solving a family of single output target (each element of the target matrix $\boldsymbol{H}$) optimization problems

$$I_{i,j}(\mathcal{U}, \mathcal{V}) = \int_0^T u^2(i,j)\mathrm{d}t + \int_0^T v^2(i,j)\mathrm{d}t. \tag{55}$$

We can obtain the following optimal index value of each single output target optimization problem by solving HJB equations

$$\frac{2}{\sigma_1(\boldsymbol{\Phi})}|H_{ij}|. \tag{56}$$

Hence, the optimal index value (DIC) of the whole optimization problem is

$$J^*(\boldsymbol{H}) = \frac{2}{\sigma_1(\boldsymbol{\Phi})}\sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j |H_{ij}|. \tag{57}$$

However, due to the competition of games themselves, the direct communication between players in the game is not realistic. Generally speaking, DIC is larger than (57); that is, the DIC of the extreme scenario is not achievable. In practice, although one player do not have prior information of the other player's action, they can communicate partial information on their actions to each other without disclosing complete information on their respective actions, by a certain protocol, so that DIC is reduced when the goal of the cooperative game is achieved. In the next section, we will use relevant results in [31] to discuss this two-phase protocol.

5 Two-phase protocol

We consider a special two-phase protocol, in one phase of which it is assumed that there is a negligible cost of transmitting partial information on the actions between the players. This can be realized by the ε-signaling capability in [31]. The main result here is that if the energy cost of transmitting partial information on the actions is negligible, the control cost (DIC) can be made arbitrarily close to the lower bound

$$\widetilde{J^*}(\boldsymbol{H}) = \frac{2\min_{i,j}\sqrt{p_i q_j}}{\sigma_1(\boldsymbol{\Phi})}\sum_{i=1}^{m}\sum_{j=1}^{n}\sqrt{p_i q_j}|H_{ij}|. \tag{58}$$

The protocol consists of two phases. In the first phase, the players communicate partial information on their actions to each other at negligible energy cost. Based on the information received, the original target matrix $\boldsymbol{H}$ is decomposed into K sub-matrices $\boldsymbol{H}_1, \ldots, \boldsymbol{H}_K$, and then, the players select the sub-matrix corresponding to their actions. In the second phase, apply the protocol in the previous sections to realize the selected sub-matrix; then, achieve the control objective.

Remark 9. The decomposition here is actually a partition; that is, the elements of different sub-matrices differ from each other and each element of the original matrix $\boldsymbol{H}$ must belong to some sub-matrix.

Remark 10. Just as in the previous section, the game requires that the players do not have to completely communicate their actions to each other.

Remark 11. The advantage of this two-phase protocol is the reduction of the control energy cost (DIC); however, there are also obvious deficiencies: the amount of communication data increases, DIC increases in another sense, and the security of the players' actions is also reduced. Hence, in practice, this protocol is applicable only when the focus is on saving energy during decision-implementation.

The sub-matrix $\boldsymbol{H}_k (k = 1, \ldots, K)$ is an $m_k \times n_k$ matrix with its (i, j) entry defined by

$$H_k(i, j) = H_{t_{i,k}, s_{j,k}}, \tag{59}$$

where $t_{i,k} \in \mathcal{M}_k$ is a subset of $\{1, \ldots, m\}$, $t_{j,k} \in \mathcal{N}_k$ is a subset of $\{1, \ldots, n\}$, and $|\mathcal{M}_k| = m_k, |\mathcal{N}_k| = n_k$.

The above sub-matrix partition can be regarded as a decomposition of the original problem (3) and (4) into K simpler sub-problems. Then, the optimal index value (DIC) of the original problem is the weighted average over the optimal index values of the decomposed subproblems; namely

$$A = \sum_{k=1}^{K} \left(\sum_{i=1}^{m_k} \sum_{j=1}^{n_k} p_{t_{i,k}} q_{s_{j,k}} \right) \hat{C}_\Phi(\boldsymbol{H}_k), \tag{60}$$

the weight

$$\sum_{i=1}^{m_k} \sum_{j=1}^{n_k} p_{t_{i,k}} q_{s_{j,k}} \tag{61}$$

is the total probability of strategy combinations in the sub-matrix $\boldsymbol{H}_k$.

The following result shows that the lower bound for A is (58).

Theorem 2. Under the conditions of Theorem 1, the optimal index value (DIC) A of the original problems (3) and (4), for applying the above two-phase protocol with the sub-matrix partition $\boldsymbol{H}_1, \ldots, \boldsymbol{H}_K$ of the target matrix $\boldsymbol{H}$, satisfies

$$A \geqslant \frac{2 \min_{i,j} \sqrt{p_i q_j}}{\sigma_1(\boldsymbol{\Phi})} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{p_i q_j} |H_{ij}|. \tag{62}$$

Remark 12. Different from the result in [31], the lower bound here is not (57) (smaller than (57) in general cases). Intuitively, Eq. (57) can be regarded as the DIC of the protocol directly communicating the complete prior information of each player's actions to each other, which is the optimal index value of the optimization problem with fewer constraints than the original problem. Hence, Eq. (57) should be the lower bound in Theorem 2. The control cost A can be close to this lower bound so that the objective of reducing DIC is achieved.

Proof. Denote

$$l_k = \min\{m_k, n_k\}, \quad k = 1, \ldots, K, \tag{63}$$

$$\boldsymbol{P}_k = \operatorname{diag} \left\{ \sqrt{\frac{p_{t_{i,k}} \sum_{j=1}^{n_k} q_{s_{j,k}}}{\sum_{i=1}^{m_k} \sum_{j=1}^{n_k} p_{t_{i,k}} q_{s_{j,k}}}} \right\}_{i=1}^{m_k}, \quad \boldsymbol{Q}_k = \operatorname{diag} \left\{ \sqrt{\frac{q_{t_{j,k}} \sum_{i=1}^{m_k} p_{s_{i,k}}}{\sum_{i=1}^{m_k} \sum_{j=1}^{n_k} p_{t_{i,k}} q_{s_{j,k}}}} \right\}_{j=1}^{n_k}, \tag{64}$$

$$\widetilde{\boldsymbol{P}_k} = \operatorname{diag} \left\{ \sqrt{p_{t_{i,k}} \sum_{j=1}^{n_k} q_{s_{j,k}}} \right\}_{i=1}^{m_k}, \quad \widetilde{\boldsymbol{Q}_k} = \operatorname{diag} \left\{ \sqrt{q_{t_{j,k}} \sum_{i=1}^{m_k} p_{s_{i,k}}} \right\}_{j=1}^{n_k}. \tag{65}$$

By the properties of eigenvalues and the matrix trace

$$\sum_{l=1}^{l_k} \sigma_l^2(\widetilde{\boldsymbol{P}_k} \boldsymbol{H}_k \widetilde{\boldsymbol{Q}_k}) = \operatorname{tr}(\widetilde{\boldsymbol{P}_k} \boldsymbol{H}_k \widetilde{\boldsymbol{Q}_k})(\widetilde{\boldsymbol{P}_k} \boldsymbol{H}_k \widetilde{\boldsymbol{Q}_k})'$$

$$= \sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\left(p_{t_{i,k}}\sum_{j=1}^{n_k}q_{s_{j,k}}\right)H_k^2(i,j)\left(q_{s_{j,k}}\sum_{i=1}^{m_k}p_{t_{i,k}}\right). \tag{66}$$

Let $x_1,\ldots,x_n$ be arbitrary real numbers. Then

$$n\sum_{i=1}^{n}x_i^2 \geqslant \left(\sum_{i=1}^{n}x_i\right)^2. \tag{67}$$

Along with (66) and Theorem 1, this gives

$$\begin{aligned}
A &= \sum_{k=1}^{K}\left(\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}p_{t_{i,k}}q_{s_{j,k}}\right)\hat{C}_{\Phi}(\boldsymbol{H}_k)\\
&= 2\sum_{k=1}^{K}\left(\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}p_{t_{i,k}}q_{s_{j,k}}\right)\sum_{l=1}^{l_k}\frac{\sigma_l(\boldsymbol{P}_k\boldsymbol{H}_k\boldsymbol{Q}_k)}{\sigma_l(\boldsymbol{\Phi})}\\
&\geqslant \frac{2}{\sigma_1(\boldsymbol{\Phi})}\sum_{k=1}^{K}\left(\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}p_{t_{i,k}}q_{s_{j,k}}\right)\sum_{l=1}^{l_k}\sigma_l(\boldsymbol{P}_k\boldsymbol{H}_k\boldsymbol{Q}_k)\\
&= \frac{2}{\sigma_1(\boldsymbol{\Phi})}\sum_{k=1}^{K}\sum_{l=1}^{l_k}\sigma_l(\widetilde{\boldsymbol{P}_k}\boldsymbol{H}_k\widetilde{\boldsymbol{Q}_k})\\
&\geqslant \frac{2}{\sigma_1(\boldsymbol{\Phi})}\sum_{k=1}^{K}\left(\sum_{l=1}^{l_k}\sigma_l^2(\widetilde{\boldsymbol{P}_k}\boldsymbol{H}_k\widetilde{\boldsymbol{Q}_k})\right)^{\frac{1}{2}}\\
&= \frac{2}{\sigma_1(\boldsymbol{\Phi})}\sum_{k=1}^{K}\left(\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\left(p_{t_{i,k}}\sum_{j=1}^{n_k}q_{s_{j,k}}\right)H_k^2(i,j)\left(q_{s_{j,k}}\sum_{i=1}^{m_k}p_{t_{i,k}}\right)\right)^{\frac{1}{2}}\\
&\geqslant \frac{2\min_{i,j}\sqrt{p_iq_j}}{\sigma_1(\boldsymbol{\Phi})}\sum_{k=1}^{K}\left(\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}n_kp_{t_{i,k}}H_k^2(i,j)m_kq_{s_{j,k}}\right)^{\frac{1}{2}}\\
&= \frac{2\min_{i,j}\sqrt{p_iq_j}}{\sigma_1(\boldsymbol{\Phi})}\sum_{k=1}^{K}\left(m_kn_k\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}(\sqrt{p_{t_{i,k}}q_{s_{j,k}}}|H_k(i,j)|)^2\right)^{\frac{1}{2}}\\
&\geqslant \frac{2\min_{i,j}\sqrt{p_iq_j}}{\sigma_1(\boldsymbol{\Phi})}\sum_{k=1}^{K}\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\sqrt{p_{t_{i,k}}q_{s_{j,k}}}|H_k(i,j)|\\
&= \frac{2\min_{i,j}\sqrt{p_iq_j}}{\sigma_1(\boldsymbol{\Phi})}\sum_{i=1}^{m}\sum_{j=1}^{n}\sqrt{p_iq_j}|H_{ij}|.
\end{aligned} \tag{68}$$

Theorem 2 is proved.

When $p_i=\frac{1}{m}, q_j=\frac{1}{n}(m=1,\ldots,m, n=1,\ldots,n)$, the lower bound in the above theorem becomes

$$\frac{2}{mn\sigma_1(\boldsymbol{\Phi})}\sum_{i=1}^{m}\sum_{j=1}^{n}|H_{ij}|. \tag{69}$$

This is precisely as in (57) and the same as the result in [31].

6 Example analysis

In this section, we use the example in Subsection 2.3 to show that the result of using Theorem 1 to calculate DIC is the same as the result of the exhaust algorithm; however, the calculation is much simpler.

For the case discussed in Subsection 2.3, the model of the example in Subsection 2.2 becomes

$$\begin{cases} x_{k+1} = x_k + \boldsymbol{u}_k'\boldsymbol{u}_k + \boldsymbol{v}_k'\boldsymbol{v}_k, \\ y_k^A = h_k^A(x_k), \\ y_k^B = h_k^B(x_k), \\ \boldsymbol{u}_k = \boldsymbol{P}(Q(y_k^A), Q(y_{k-1}^A), \alpha), \\ \boldsymbol{v}_k = \boldsymbol{P}(Q(y_k^B), Q(y_{k-1}^B), \beta), \\ z_k = I_{\{x_k \geqslant 5\}}, \end{cases} \tag{70}$$

with DIC index

$$I(\mathcal{U}, \mathcal{V}) = \sum_{i=1}^{2}\sum_{j=1}^{2} p_i q_j \left(\sum_{k=0}^{T-1} \boldsymbol{u}_k(i)'\boldsymbol{u}_k(i) + \sum_{k=0}^{T-1} \boldsymbol{v}_k(j)'\boldsymbol{v}_k(j) \right). \tag{71}$$

In the calculation performed in Subsection 2.3, by specific control measures $\boldsymbol{u}_k(\alpha), \boldsymbol{v}_k(\beta)$, we have obtained the energy cost for the control measures corresponding to each strategy; thereby, we obtained the DICs of the three Nash equilibrium points. The DIC of the mixed strategy Nash equilibrium point(σ^*, σ^*) with $\sigma^* = \{1/3, 2/3\}$ was calculated as $\frac{16}{9}T + 2$.

In the following, we will use Theorem 1 to calculate the DIC of the mixed strategy Nash equilibrium point (σ^*, σ^*) in Subsection 2.3 and show that this calculation is simpler than the calculation performed in Subsection 2.3. Then, we will analyze the result.

Take an orthonormal base of the admissible control set $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_{2T}\}$, satisfying

$$\boldsymbol{\xi}_i = \{\boldsymbol{\xi}_k^i\}_{k=0}^{T-1}, \quad \boldsymbol{\xi}_k^i = \begin{cases} (0,1)', & k = (i-1)/2,\ i \text{ is odd}, \\ (1,0)', & k = i/2 - 1,\ i \text{ is even}, \\ (0,0)', & \text{others}, \end{cases} \quad i = 1, \ldots, 2T. \tag{72}$$

By simple analysis and calculation

$$\boldsymbol{P} = \boldsymbol{Q} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \\ & \sqrt{\frac{2}{3}} \end{pmatrix}, \quad \boldsymbol{H} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{\Phi} = \begin{pmatrix} \frac{1}{2T-1} & & & & \\ & \frac{9}{8T+9} & & & \\ & & \frac{1}{2T-1} & & \\ & & & \ddots & \\ & & & & \frac{1}{2T-1} \end{pmatrix}_{2T \times 2T}. \tag{73}$$

From the analysis in Remark 6, $T > 2$; namely, $T \geqslant 3$. Hence, by Theorem 1, DIC is

$$\hat{C}_{\Phi}(\boldsymbol{H}) = 2 \sum_{k=1}^{\min(m,n)} \frac{\sigma_k(\boldsymbol{P}HQ)}{\sigma_k(\boldsymbol{\Phi})} = 2 \cdot \frac{1}{\frac{9}{8T+9}} = \frac{16}{9}T + 2. \tag{74}$$

This result is the same as the result obtained in Subsection 2.3. Moreover, since we only need to calculate $\boldsymbol{\Phi}$, the calculation of DIC is much simpler than the calculation performed in Subsection 2.3. This is also an advantage of Theorem 1. The calculations for the DICs of the other two Nash equilibrium points are similar.

It can be seen that the DICs of the three Nash equilibrium points differ significantly. Since $T \geqslant 3$, the DIC of the mixed strategy Nash equilibrium point (σ^*, σ^*) is no less than $\frac{22}{3}$; therefore, the DICs of the Nash equilibrium points (1,2) and (2,1) are smaller than that of (σ^*, σ^*). In this example, the CDMs of the three Nash equilibrium points are the same (the time complexities of the algorithms are the same). Hence, from Definition 1, the DCs of the Nash equilibrium points (①,②) and (②,①) are much smaller than that of the mixed strategy Nash equilibrium point (σ^*, σ^*); therefore, the two pure strategy Nash equilibrium points should be preferred. However, it is interesting that the mixed strategy Nash equilibrium (σ^*, σ^*) is an evolutionary stable strategy [34]; therefore, it has more theoretical advantages than the other two pure strategy Nash equilibrium points. This implies that it is relatively new and more practically meaningful in choosing the DC as the criterion for selecting optimal strategies.

7 Concluding remarks

In this paper, we studied the DIC of cooperative games from an energy perspective. The definition of DC was proposed, the decision-implementation system model of cooperative games and an energy-based index were established, a definition of DIC was presented, and an analytical expression of DIC was obtained. Moreover, a two-phase protocol was designed to reduce DIC, with partial information on the actions shared.

The explicit relationships between practically meaningful DIC indicators (such as the amount of communication data, time complexity, and decision information security) and control protocols, and the optimization problems still require further study. For the main result (38), the specific algorithm to calculate the DICs of real systems needs to be designed further. At the same time, by adding controllability and stability of systems, the complexity of the issue will increase. Moreover, in this paper, only cooperative games with two players and Nash equilibriums were discussed. The cases of multiple players (possibly related to topology problems), other different equilibriums, non-cooperative games, and dynamic decision processes, require further study. Furthermore, the topic of this paper can be extended to applications of game theory in wireless communications [35, 36].

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References

1 de Deus T F, Lopes P F. A game about biology for biology students cell life as a learning tool. In: Proceedings of the 8th Iberian Conference on Information Systems and Technologies, Lisboa, 2013. 1–6
2 Chen L, Yu N, Su Y. The application of game theory in the Hercynian economic development. In: Proceedings of International Conference on Management Science and Engineering, Helsinki, 2014. 794–799
3 Ott U F. International business research and game theory: looking beyond the prisoner's dilemma. Int Bus Rev, 2013, 22: 480–491
4 Chou H M, Zhou L. A game theory approach to deception strategy in computer mediated communication. In: Proceedings of IEEE International Conference on Intelligence and Security Informatics, Arlington, 2012. 7–11
5 Abellanas M, López M D, Rodrigo J. Searching for equilibrium positions in a game of political competition with restrictions. Eur J Oper Res, 2010, 201: 892–896
6 Yao P, Jiang D, Zhu T. On optimization of auction mechanism in the military logistics based on the game theory. In: Proceedings of the 2nd International Conference on Information Science and Engineering, Hangzhou, 2010. 5940–5943
7 Liu T F, Lu X S, Jiang Z P. A junction-by-junction feedback-based strategy with convergence analysis for dynamic traffic assignment. Sci China Inf Sci, 2016, 59: 010203
8 Qi L Y, Hu X M. Design of evacuation strategies with crowd density feedback. Sci China Inf Sci, 2016, 59: 010204
9 Sasaki Y, Kijima K. Hierarchical hypergames and Bayesian games: a generalization of the theoretical comparison of hypergames and Bayesian games considering hierarchy of perceptions. J Syst Sci Complex, 2016, 29: 187–201
10 Yuan Y. A dynamic games approach to H^∞ control design of DoS with application to longitudinal flight control. Sci China Inf Sci, 2015, 58: 092202
11 Mesmer B L, Bloebaum C L. Incorporation of decision, game, and Bayesian game theory in an emergency evacuation exit decision model. Fire Safety J, 2014, 67: 121–134
12 Qi X, Zhang Y, Meng J. Game theory model and equilibrium analysis of peasant's production decision. J Northeast Agr Univ (English Edition), 2012, 19: 71–73
13 Bando K. On the existence of a strictly strong Nash equilibrium under the student-optimal deferred acceptance algorithm. Games Econ Behav, 2014, 87: 269–287

14 Dai Y, Yang H, Wu J, et al. The joint non-inferior Nash equilibrium analysis of electricity futures and spot market. In: Proceedings of Transmission and Distribution Conference and Exposition: Asia and Pacific, Seoul, 2009. 1–4

15 Zhang C, Gao J, Liu Z, et al. Nash equilibrium strategy of two-person bilinear-quadratic differential game: a recursive approach. Intell Control Automat, 2006, 1: 1075–1079

16 Sintchenko V, Coiera E. Decision complexity affects the extent and type of decision support use. AMIA Annu Symp Proc, 2006, 2006: 724–728

17 Dabbagh S R, Sheikh-El-Eslami M K. Risk-based profit allocation to DERs integrated with a virtual power plant using cooperative game theory. Electr Power Syst Res, 2015, 121: 368–378

18 Lozano S. Information sharing in DEA: a cooperative game theory approach. Eur J Oper Res, 2012, 222: 558–565

19 Tao Z, Liu X, Chen H, et al. Group decision making with fuzzy linguistic preference relations via cooperative games method. Comput Ind Eng, 2015, 83: 184–192

20 Rashedi N, Kebriaei H. Cooperative and non-cooperative Nash solution for linear supply function equilibrium game. Appl Math Comput, 2014, 244: 794–808

21 Yao A C C. Some complexity questions related to distributive computing. In: Proceedings of the 11th Annual ACM Symposium on Theory of Computing, Atlanta, 1979

22 Impagliazzo R, Williams R. Communication complexity with synchronized clocks. In: Proceedings of IEEE 25th Annual Conference on Computational Complexity, Cambridge, 2010. 259–269

23 Klauck H. Rectangle size bounds and threshold covers in communication complexity. In: Proceedings of the 18th IEEE Annual Conference on Computational Complexity, Aarhus, 2003. 118–134

24 Kushilevitz E, Nisan N. Communication complexity. Sci Comput Program, 1999, 33: 215–216

25 Shigeta M, Amano K. Ordered biclique partitions and communication complexity problems. Discrete Appl Math, 2015, 184: 248–252

26 Baillieul J, Wong W S. The standard parts problem and the complexity of control communication. In: Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, 2009. 2723–2728

27 Wong W S. Control communication complexity of distributed control systems. SIAM J Control Optimiz, 2009, 48: 1722–1742

28 Wong W S, Baillieul J. Control communication complexity of nonlinear systems. Commun Inf Syst, 2009, 9: 103–140

29 Baillieul J, Özcimder K. The control theory of motion-based communication: problems in teaching robots to dance. Am Control Conf, 2011, 50: 4319–4326

30 Yao Y, Gehrke J. Query processing for sensor networks. In: Proceedings of the 1st Biennial Conference in Innovative Asta Systems Research (CIDR), Asilomar, 2003. 1364

31 Wong W S, Baillieul J. Control communication complexity of distributed actions. IEEE Trans Automat Control, 2012, 57: 2731–2745

32 Liu Z, Wong W S. Choice-based cluster consensus in multi-agent systems. In: Proceedings of the 32nd Chinese Control Conference, Xi'an, 2013. 7285–7290

33 Liu Z, Wong W S, Guo G. Cooperative control of linear systems with choice actions. In: Proceedings of American Control Conference, Washington, 2013. 5374–5379

34 Wang Y H, Cheng D Z. Dynamics and stability for a class of evolutionary games with time delays in strategies. Sci China Inf Sci, 2016, 59: 092209

35 Zhang H, Jiang C, Beaulieu N C, et al. Resource allocation for cognitive small cell networks: a cooperative bargaining game theoretic approach. IEEE Trans Wirel Commun, 2015, 14: 3481–3493

36 Zheng W, Su T, Zhang H, et al. Distributed power optimization for spectrum-sharing femtocell networks: a fictitious game approach. J Netw Comput Appl, 2014, 37: 315–322